**Inequalities 2**

**1.** Solve $\frac{x^{3}-3x^{2}+x+1}{x^{3}+2x^{2}+3x+2}\leq 0$

 $\frac{\left(x-1\right) \left(x^{2}-2 x-1\right)}{\left(x+1\right) \left(x^{2}+x+2\right)}\leq 0$

 Since $x^{2}+x+2=\left(x+\frac{1}{2}\right)^{2}+\frac{7}{4}>0$ for all $x\in R$

 The given inequality is reduced to $\frac{\left(x-1\right) \left(x^{2}-2 x-1\right)}{x+1}\leq 0$

 $\left(x+1\right)^{2}\frac{\left(x-1\right) \left(x^{2}-2 x-1\right)}{x+1}\leq 0$ and $x\ne 1$

 Therefore for $x\ne 1$,

$$\left(x+1\right)\left(x-1\right) \left(x^{2}-2 x-1\right)\leq 0$$

 $\left\{\begin{array}{c}\left(x+1\right)\left(x-1\right)\leq 0\\\left[x-\left(1-\sqrt{2}\right)\right]\left[x-\left(1+\sqrt{2}\right)\right]\geq 0\end{array}\right.$ or $\left\{\begin{array}{c}\left(x+1\right)\left(x-1\right)\geq 0\\\left[x-\left(1-\sqrt{2}\right)\right]\left[x-\left(1+\sqrt{2}\right)\right]\leq 0\end{array}\right.$ $\left\{\begin{array}{c}-1\leq x \leq 1\\1-\sqrt{2}\geq x or x\geq 1+\sqrt{2}\end{array}\right.$ or $\left\{\begin{array}{c}-1\geq x or x\geq 1\\1-\sqrt{2}\leq x\leq 1+\sqrt{2}\end{array}\right.$ $-1\leq x \leq 1-\sqrt{2}$ or $1\leq x\leq 1+\sqrt{2}$

 Since $x\ne 1$, the complete solution is

 $-1<x \leq 1-\sqrt{2}$ or $1\leq x\leq 1+\sqrt{2}$.

**2.** Solve $\cos(2θ)>3 sin θ+2$ for $θ$ , where $-π<θ<π$ .

 $1-2 sin^{2} θ>3 sin θ+2$

 $2 sin^{2} θ+3 sin θ+1<0$

 $\left(sin θ+1\right)\left(2 sin θ+1\right)<0$

 $-1<sin θ<-\frac{1}{2}$

 If $-π<θ<π$ ,

 the roots of $sin θ=-1$ is $θ=-\frac{π}{2}$

 and $sin θ=-\frac{1}{2}$ are $θ=-\frac{5π}{6},-\frac{π}{6}$

 ∴ The solution is $-\frac{5π}{6}<θ<-\frac{π}{2} or -\frac{π}{2}<θ<-\frac{π}{6}$ .

**3.** If $a,b,c\geq 0$, use A.M. $\geq $ G.M., or otherwise, show that

 $\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\geq \frac{3}{2}$ .

 **Method 1**

 $\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}=\left(\frac{a+b+c}{a+b}-1\right)+\left(\frac{a+b+c}{b+c}-1\right)+\left(\frac{a+b+c}{c+a}-1\right)$

 $=\frac{a+b+c}{a+b}+\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}-3=\left(a+b+c\right)\left[\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right]=3$

 $\geq \frac{3\left(a+b+c\right)}{\left[\left(a+b\right)\left(b+c\right)\left(c+a\right)\right]^{3}}-3$ (A.M. $\geq $ G.M)

 $\geq \frac{3\left(a+b+c\right)}{\frac{1}{3}\left[\left(a+b\right)+\left(b+c\right)+\left(c+a\right)\right]}-3$ (A.M. $\geq $ G.M)

 $=\frac{3\left(a+b+c\right)}{\frac{2}{3}\left(a+b+c\right)}-3=\frac{3}{2}$

 **Method 2**

 By CBS inequality (Cauchy – Bunyakovskii – Schwarz inequality)

 $\left[\left(a+b\right)+\left(b+c\right)+\left(c+a\right)\right]\left[\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right]\geq \left(1+1+1\right)^{2}$

 $2\left(a+b+c\right)\left[\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right]\geq 9$

 $\frac{a+b+c}{a+b}+\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}\geq \frac{9}{2}$

 $\frac{c}{a+b}+1+\frac{a}{b+c}+1+\frac{b}{c+a}+1\geq \frac{9}{2}$

 $\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\geq \frac{3}{2}$

 Equality holds $⟺\frac{a+b}{\frac{1}{a+b}}=\frac{b+c}{\frac{1}{b+c}}=\frac{c+a}{\frac{1}{c+a}}⟺\left(a+b\right)^{2}=\left(b+c\right)^{2}=\left(c+a\right)^{2}⟺a=b=c$

 (Given : $a,b,c\geq 0$)

**4.** If $a,b,c\in R$ and $a+b+c=2$, show that $a^{2}+b^{2}+c^{2}\geq \frac{4}{3}$.

 Find the condition for the equality.

 **Method 1**

 Use CBS inequality (Cauchy – Bunyakovskii – Schwarz inequality) for the set

 $a,b,c ; 1,1,1$

 $\left(a^{2}+b^{2}+c^{2}\right)\left(1^{2}+1^{2}+1^{2}\right)\geq \left(a+b+c\right)^{2}=2^{2}=4$

 Therefore, $a^{2}+b^{2}+c^{2}\geq \frac{4}{3}$.

 Equality holds $⟺\frac{a}{1}=\frac{b}{1}=\frac{c}{1}⟺a=b=c=\frac{2}{3}$

 **Method 2**

 $a+b+c=2$

 $a^{2}+b^{2}+c^{2}+2ab+2bc+2ca=4$

 $3\left(a^{2}+b^{2}+c^{2}\right)=4+\left(a-b\right)^{2}+\left(b-c\right)^{2}+\left(c-a\right)^{2}$

 $\geq 4$

 $a^{2}+b^{2}+c^{2}\geq \frac{4}{3}$ Equality holds $⟺a=b=c=\frac{2}{3}$

**5.** Given that $a,b,c,d$ are real numbers and $\left\{\begin{array}{c}a+b+c+d=6\\a^{2}+b^{2}+c^{2}+d^{2}=12\end{array}\right.$

 find the maximum value of $d$ .

 **Method 1**

 By CBS inequality, $\left(1^{2}+1^{2}+1^{2}\right)\left(a^{2}+b^{2}+c^{2}\right)\geq \left(a+b+c\right)^{2}$

 $3\left(12-d^{2}\right)\geq \left(6-d\right)^{2}$

 $36-3d^{2}\geq 36-12d+d^{2}$

 $4d^{2}-12d\leq 0$

 $d\left(d-3\right)\leq 0$

 $0\leq d\leq 3$, hence the maximum of $d$ is 3.

 **Method 2**

 $6-d=a+b+c$

 $\left(6-d\right)^{2}=a^{2}+b^{2}+c^{2}+2ab+2bc+2ca$

 $=3\left(a^{2}+b^{2}+c^{2}\right)-\left(a-b\right)^{2}-\left(b-c\right)^{2}-\left(c-a\right)^{2}\leq 3\left(a^{2}+b^{2}+c^{2}\right)$

 $=3\left(12-d^{2}\right)$

 Hence, $3\left(12-d^{2}\right)\geq \left(6-d\right)^{2}$

 $36-3d^{2}\geq 36-12d+d^{2}$

 $4d^{2}-12d\leq 0$

 $d\left(d-3\right)\leq 0$

 $0\leq d\leq 3$, hence the maximum of $d$ is 3.

**6.** Solve $\left|\frac{4}{x-1}\right|\geq 3\left(1-\frac{1}{x}\right)=3\left(\frac{x-1}{x}\right)$

 Obviously, $x\ne 1 or 0$.

 **(a)** If $x>1$, then
 $\frac{4}{x-1}\geq 3\left(\frac{x-1}{x}\right)$

 $\frac{4}{x-1}-3\left(\frac{x-1}{x}\right)\geq 0$

 $\frac{4x-3(x-1)^{2}}{x\left(x-1\right)}\geq 0$

 $\frac{-3 x^{2}+10 x-3}{x\left(x-1\right)}\geq 0$

 $\frac{3 x^{2}-10 x+3}{x\left(x-1\right)}\leq 0$

 $\frac{\left(3x-1\right)(x-3)}{x\left(x-1\right)}\leq 0$

 $0<x\leq \frac{1}{3} or 1<x\leq 3$

 But $x>1$, therefore $1<x\leq 3$

 **(b)** If $x<1$, then
 $-\frac{4}{x-1}\geq 3\left(\frac{x-1}{x}\right)$

 $3\left(\frac{x-1}{x}\right)+\frac{4}{x-1}\leq 0$

 $\frac{3(x-1)^{2}+4x}{x\left(x-1\right)}\leq 0$

 $\frac{3 x^{2}-2 x+3}{x\left(x-1\right)}\leq 0$

 For $3 x^{2}-2 x+3$ , $∆\leq 0$

 Therefore, $3 x^{2}-2 x+3\geq 0$ for all x.

 The inequality become $\frac{1}{x\left(x-1\right)}\leq 0$

 Solving, $0\leq x\leq 1$

 But $x<1 and x\ne 0$,

 Therefore $0<x<1.$

 Joining (a) and (b), $0<x<1 or 1<x\leq 3$.

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